Triviality of GHZ operators of higher spin

2000 J. Phys. A: Math. Gen. 33 L493

(http://iopscience.iop.org/0305-4470/33/48/105)

J. Phys. A: Math. Gen. **33** (2000) L493–L495. Printed in the UK PII: S0305-4470(00)15788-9

LETTER TO THE EDITOR

Triviality of GHZ operators of higher spin

J Savinien, J Taron and R Tarrach

Departament d'Estructura i Constituents de la Matèria, Facultat de Física, Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain, and Institut de Fisica d'Altes Energies

E-mail: taron@ecm.ub.es

Received 21 July 2000

Abstract. We prove that local observables of the set of GHZ operators for particles of spin higher than $\frac{1}{2}$ reduce to direct sums of the spin-$\frac{1}{2}$ operators σ_x and σ_y and, therefore, no new contradictions with local realism arise by considering them.

1. Introduction

The GHZ theorem [1] provides a powerful test of quantum non-locality, which can be confirmed or refuted by the outcome of just one experiment [2]. Formulated for three spin-$\frac{1}{2}$ particles [2,3], the argument is based on the anti-commutative nature of the 2×2 spin operators σ_x and σ_y. The values of the three mutually commuting observables

$$\sigma_x^a \otimes \sigma_y^b \otimes \sigma_y^c \equiv \sigma_x^a \sigma_y^b \sigma_y^c \qquad \sigma_y^a \sigma_x^b \sigma_y^c \qquad \sigma_y^a \sigma_y^b \sigma_x^c \tag{1}$$

and their product, $-\sigma_x^a \sigma_x^b \sigma_x^c$, cannot be obtained, consistently, by making local assignments to each of the individual spin operators, $m_x^I, m_y^I = \pm 1$, $I = a, b, c$. This is not a contradiction of quantum mechanics: the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle)$, for instance, is one of the common eigenstates of the four operators, with eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\lambda_4 = -1$, respectively. $|\psi\rangle$ is a highly correlated (entangled) state of the three parties, which has no defined value for σ_x^I, σ_y^I.

In this letter we address the question of how to generalize the argument to particles of higher spin and find that there are no non-trivial extensions other than direct sums of operators that can be brought into the form σ_x, σ_y by means of local unitarity transformations. (For odd-dimensional Hilbert spaces the direct sum is completed by a one-dimensional submatrix, i.e. a c-number in the diagonal.) We give a proof for the cases of spin 1 and $\frac{3}{2}$. Similar problems have been addressed in [4].

Let us look for observables A and B such that $AB = \omega BA$ (their hermiticity implies that ω is at most a phase): this is a necessary condition for the commutator relations $[A_1^a A_2^b A_3^c, B_1^a B_2^b B_3^c] = \text{etc} \ldots = 0$ to hold. As we shall see, all interesting cases correspond to $\omega = -1$. Without loss of generality, A can always be taken as diagonal, $A = \text{diag}(\lambda_1, \lambda_2)$, for the simplest case $s = \frac{1}{2}$. The above condition reads

$$AB - \omega BA = \begin{pmatrix} (1-\omega)\lambda_1 b_{11} & (\lambda_1 - \omega\lambda_2) b_{12} \\ (\lambda_2 - \omega\lambda_1) b_{12}^* & (1-\omega)\lambda_2 b_{22} \end{pmatrix} = 0. \tag{2}$$

If $\omega \neq 1$, a solution with non-vanishing off-diagonal elements is allowed if $\omega^2 = 1$, i.e. $\omega = -1$. This leads to

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix} \tag{3}$$

which can always be transformed to σ_x and σ_y, by rotations and adequate normalization. These are the operators of the example (1). For spin $\frac{1}{2}$ the set of GHZ operators are in this sense unique.

2. Spin 1

For higher spins the proof proceeds along the same lines. We find one case of interest, with $\omega = -1$,

$$A = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & b & c \\ b^* & 0 & 0 \\ c^* & 0 & 0 \end{pmatrix}. \tag{4}$$

In the basis where B is diagonal, A and B read as

$$A = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \sqrt{|b|^2 + |c|^2}\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \tag{5}$$

which proves the assertion in the case of spin 1, as a rotation around x brings B into the form $0 \oplus \sigma_y$, while A is left as $1 \oplus \sigma_x$, up to normalizations.

3. Spin $\frac{3}{2}$

For spin $\frac{3}{2}$, in addition to cases that reduce straightforwardly to those of lower spins, we find

$$A = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & a & b & c \\ a^* & 0 & 0 & 0 \\ b^* & 0 & 0 & 0 \\ c^* & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

In the basis where B is diagonal, A and B read as

$$A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \sqrt{|a|^2 + |b|^2 + |c|^2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \tag{7}$$

which is again diagonal in two 2×2 blocks.

The last case corresponds to

$$A = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & a & 0 & b \\ a^* & 0 & c^* & 0 \\ 0 & c & 0 & d \\ b^* & 0 & d^* & 0 \end{pmatrix}. \tag{8}$$

The following list of unitary transformations brings these matrices to the desired form.

(a) With

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{9}$$

$F^\dagger = F = F^{-1}$, we find

$$A' = FAF = \begin{pmatrix} I & \\ & -I \end{pmatrix} \qquad B' = FBF = \begin{pmatrix} & \mathcal{B} \\ \mathcal{B}^\dagger & \end{pmatrix} \tag{10}$$

where

$$\mathcal{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(b) A unitary transformation of the form $U = \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix}$ leaves A' invariant and allows us to diagonalize $\mathcal{B}$

$$A'' = A' \qquad B'' = UB'U^\dagger = \begin{pmatrix} & U_1\mathcal{B}U_2^\dagger \\ (U_1\mathcal{B}U_2^\dagger)^\dagger & \end{pmatrix} = \begin{pmatrix} & & m & 0 \\ & & 0 & n \\ m^* & 0 & & \\ 0 & n^* & & \end{pmatrix}. \tag{11}$$

We have used the result that the generic matrix $\mathcal{B}$ can be brought to a diagonal form with two unitary matrices U_1 and U_2.

(c) Finally, acting with F again,

$$A''' = A \qquad B''' = \begin{pmatrix} 0 & m & & \\ m^* & 0 & & \\ & & 0 & n \\ & & n^* & 0 \end{pmatrix} \tag{12}$$

which completes the proof.

4. Conclusions

We conclude that the equation $AB = \omega BA$ is very restrictive on ω and on the possible forms of A and B; as the Hilbert space dimension increases, with increasing spin, all its solutions for $\omega \neq 1$ have $\omega = -1$ and are essentially direct sums of the two-dimensional σ_x and σ_y. In this sense there are no solutions that could, in principle, enrich the possibilities opened by the GHZ theorem.

JS, JT and RT acknowledge the Centro de Ciencias de Benasque for hospitality while this work was being done. JS also acknowledges the ECM Department for hospitality and financial support. JT and RT acknowledge financial support by CICYT project AEN 98-0431, CIRIT project 1998 SGR-00026 and CEC project IST-1999-11053.

References

[1] Greenberger D M, Horne M and Zeilinger A 1989 Going beyond Bell's theorem *Bell's Theorem, Quantum Theory, and Conception of the Universe* ed M Kafatos (Dordrecht: Kluwer) pp 69–72

[2] Mermin N D 1990 *Am. J. Phys.* **58** 731

[3] Peres A 1995 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer)

[4] Cabello A unpublished